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On a deformation of $sl(2)$ with paragrassmannian variables

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Abstract. We propose a new structure $\mathcal{U}_q^r(sl(2))$. This is realized by multiplying $\delta(q = e^\delta$, $\delta \in \mathbb{C})$ by θ, where θ is a real nilpotent, paragrassmannian, variable of order $r(\theta^{r+1} = 0)$ that we call the order of deformation, the limit $r \to \infty$ giving back the standard $\mathcal{U}_q(sl(2))$. In particular, we show that for $r = 1$ there exists a new $\mathcal{R}$-matrix associated with $sl(2)$. We also prove that the restriction of the values of the parameters of deformation give nonlinear algebras as particular cases.

1. Introduction

During the last few years, q-deformations [1] ($q = e^\delta, \delta \in \mathbb{C}$) of the universal enveloping algebra of Lie algebras have attracted wide attention. They are indeed remarkable mathematical structures known as Hopf algebras and they have been proved to be connected to conformal field theory and to solvable model (see [2] and references therein).

The pioneering papers [3] devoted to the specific $\mathcal{U}_q(sl(2))$ case have been extended by various authors. Let us just mention here the Roček proposal [4] (based on generalized nonlinear deformations) providing a new algebraic description of the Morse and modified Pöschl–Teller Hamiltonians [5]. Despite its physical interest the Roček deformation has been rarely exploited as compared to the Drinfeld–Jimbo deformation, because of its mathematical defect: its Hopf characteristics (coproduct, counit, antipode) have not yet been pointed out.

In this paper, we answer the following question: Is it possible to obtain the nonlinear algebras as particular restrictions of the quantum deformation?

Our purpose is then twofold. First, we introduce the nilpotent algebra $\mathcal{U}_q^r(sl(2))$ by multiplying δ by θ is a real nilpotent, paragrassmannian, variable [6] of order $r(\theta^{r+1} = 0)$. Second, we discuss the connection of this new structure to some particular nonlinear deformations of $sl(2)$ whose Hopf characteristics are introduced.

In section 2, we briefly review the Drinfeld–Jimbo deformation of $sl(2)$. Then, in section 3, we introduce the quantization with one paragrassmannian variable and its Hopf structure. The quantization with two paragrassmannian variables is given in section 4. In

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section 5, we give the connection of these structures to particular nonlinear deformations of $sl(2)$. Finally, we conclude in section 6 with some comments.

2. The $\mathcal{U}_q(sl(2))$ algebra

The standard Drinfeld–Jimbo deformation [1] of the Lie algebra $sl(2)$ generated by H, J_+ and J_- is characterized by the relations

$$[J_+, J_-] = \frac{q^H - q^{-H}}{q - q^{-1}} = \frac{\sinh(\delta H)}{\sinh(\delta)} \qquad [H, J_\pm] = \pm 2 J_\pm. \tag{2.1}$$

It is completed by the following additional operations, coproduct Δ: $\mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$, counit ε: $\mathcal{U}_q(sl(2)) \to \mathbb{C}$ and the antipode S: $\mathcal{U}_q(sl(2)) \to {}_q(sl(2))$, such that

$$\begin{aligned}
&\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H \\
&\Delta(J_\pm) = J_\pm \otimes \mathrm{e}^{\delta H/2} + \mathrm{e}^{-\delta H/2} \otimes J_\pm \\
&\varepsilon(\mathbf{1}) = \mathbf{1} \qquad \varepsilon(J_\pm) = \varepsilon(H) = 0 \\
&S(\mathbf{1}) = \mathbf{1} \qquad S(H) = -H \qquad S(J_\pm) = -\mathrm{e}^{\pm\delta} J_\pm
\end{aligned} \tag{2.2}$$

where Δ and ε are homomorphisms while S is an algebra antihomomorphism

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \qquad S(ab) = S(b)S(a). \tag{2.3}$$

Moreover, if m: $\mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2))$ stands for the multiplication mapping of $\mathcal{U}_q(sl(2))$, i.e. $m(a \otimes b) = a \cdot b$, we have

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \\
&m(\mathrm{id} \otimes S)\Delta = m(S \otimes \mathrm{id})\Delta = \mathrm{i} \circ \varepsilon \\
&(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}.
\end{aligned} \tag{2.4}$$

These are just all the axioms of a Hopf algebra, and so $\mathcal{U}_q(sl(2))$ endowed with ε, Δ and S just forms a Hopf algebra.

Let us define the formal series

$$J_\pm = \sum_{k=0}^{\infty} \delta^k J_\pm^{(+)} \tag{2.5}$$

and

$$\frac{\sinh(H\delta)}{\sinh(\delta)} = \sum_{k=0}^{\infty} \psi_k(H) \delta^{2k} \tag{2.6}$$

the second formula being just the result of a Taylor expansion. The generators $J_\pm^{(k)}$ and H satisfy the following commutation relations:

$$\begin{aligned}
&[H, J_\pm^{(k)}] = \pm 2 J_\pm^{(k)} \\
&\sum_{m=0}^{2k} [J_+^{(m)}, J_-^{(2k-m)}] = \psi_k(H) \qquad \sum_{m=0}^{2k+1} [J_+^{(m)}, J_-^{(2k+1-m)}] = 0 \\
&\sum_{m=0}^{k} [J_\pm^{(m)}, J_\pm^{(k-m)}] = 0.
\end{aligned} \tag{2.7}$$

Its Hopf structure is given by

$$\Delta(H) = \mathbf{1}\otimes H + H\otimes \mathbf{1}$$
$$\Delta(J_\pm^{(k)}) = \sum_{m=0}^{k} \frac{1}{2^m m!}((-1)^m H^m \otimes J_\pm^{(k-m)} + J_\pm^{(k-m)} \otimes H^m)$$
$$\varepsilon(H) = \varepsilon(J_\pm^{(k)}) = 0 \qquad \varepsilon(\mathbf{1}) = \mathbf{1}$$
$$S(J_\pm^{(k)}) = -\sum_{n=0}^{k} \frac{(\pm)^m}{m!} J_\pm^{(k-m)} \qquad S(H) = -H \qquad S(\mathbf{1}) = \mathbf{1} \tag{2.8}$$

as can be verified.

3. The $\mathcal{U}_q^r(sl(2))$ algebra

Let us introduce the real nilpotent, paragrassmannian, variable θ of order r, i.e.

$$\theta^{r+1} = 0 \tag{3.1}$$

being realized, in a simple way, by

$$\theta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.2}$$

Besides this choice, we want to notice that there are other representations such as that given by

$$\theta^r = \sum_{\alpha=1}^{r} \theta^{(\alpha)} \tag{3.3}$$

where

$$(\theta^{(\alpha)})^2 = 0 \qquad [\theta^{(\alpha)}, \theta^{(\beta)}] = 0 \qquad \alpha \neq \beta. \tag{3.4}$$

Then with equation (3.2), we propose to generalize the operators (2.5) through

$$J_\pm^\theta = \sum_{m=0}^{r} \delta^m \theta^m J_\pm^{(m)} \tag{3.5}$$

$$= \begin{pmatrix} J_\pm^{(0)} & 0 & \cdots & 0 \\ \delta J_\pm^{(1)} & J_\pm^{(0)} & \cdots & 0 \\ \delta^{r-1} J_\pm^{(r-1)} & \ddots & \ddots & \vdots \\ \delta^r J_\pm^{(r)} & \delta^{r-1} J_\pm^{(r-1)} & \delta J_\pm^{(1)} & J_\pm^{(0)} \end{pmatrix}. \tag{3.6}$$

Using the commutation relations (2.7), we thus have

$$[H, J_\pm^\theta] = \pm 2 J_\pm^\theta \tag{3.7}$$

and

$$\begin{aligned} [J_+^\theta, J_-^\theta] &= \sum_{k=0}^{r} \delta^k \theta^k \left(\sum_{m=0}^{k} [J_+^{(m)}, J_-^{(k-m)}] \right) \\ &= \psi_0(H) + \theta^2\delta^2\psi_1(H) + \cdots + \theta^{2[r/2]}\delta^{2[r/2]}\psi_{[r/2]}(H) \\ &= \sum_{k=0}^{[r/2]} \psi_k(H)\theta^{2k}\delta^{2k} \end{aligned} \tag{3.8}$$

where $[\lambda]$ stands for the integer part of λ. Defining the exponential map by

$$e(x;\theta) = \sum_{k=0}^{r} \frac{x^k\theta^k}{k!} \tag{3.9}$$

we can finally write

$$[J_+^\theta, J_-^\theta] = \frac{e(H\delta;\theta) - e(-H\delta;\theta)}{e(\delta;\theta) - e(-\delta;\theta)} \qquad [H, J_\pm^\theta] = \pm 2J_\pm^\theta. \tag{3.10}$$

The algebra $\{J_\pm^\theta, H\}$ described by the commutation relations (3.10) is just the deformation of $sl(2)$ with one paragrassmannian variable and is denoted by $\mathcal{U}_q^r(sl(2))$. This algebra is isomorphic to $\mathcal{U}_q(sl(2))/(\delta^{r+1}\mathcal{U}_q(sl(2))$, i.e.

$$\mathcal{U}_q^r(sl(2)) \cong \mathcal{U}_q(sl(2))/(\delta^{r+1}\mathcal{U}_q(sl(2))).$$

In order to define a Hopf structure for $\mathcal{U}_q^r(sl(2))$, we need the following definition

Definition 1. Let

$$a = a_0 + a_1\theta + \cdots + a_r\theta^r \qquad b = b_0 + b_1\theta + \cdots + b_r\theta^r. \tag{3.11}$$

The tensor product between a and b is defined by

$$a\bar{\otimes}b = \sum_{m=1}^{r}\sum_{n=1}^{r} a^{(m)} \otimes b^{(n)}\theta^{m+n} \tag{3.12}$$

and

$$(a\bar{\otimes}b)(c\bar{\otimes}d) = (ac\bar{\otimes}bd). \tag{3.13}$$

This operation is called the paragrassmannian tensor product.

When the paragrassmannian order $r \to \infty$, this operation is equivalent to the standard one. This paragrassmannian tensor product is compatible with

$$\mathcal{U}_q^r(sl(2))\bar{\otimes}\mathcal{U}_q^r(sl(2)) \equiv \mathcal{U}_q^r(so(4)) \tag{3.14}$$

and with the inclusion

$$\mathcal{U}_q^r(sl(2)) \subset \mathcal{U}_q^r(sl(3)) \subset \cdots \subset \mathcal{U}_q^r(sl(N-1)) \subset \mathcal{U}_q^r(sl(N)). \tag{3.15}$$

We are now able to claim that

Proposition 1. The Hopf structure associated to the $\mathcal{U}_q^r(sl(2))$ is given by

$$\begin{aligned}
&\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H \\
&\Delta(J_\pm^\theta) = J_\pm^\theta \bar{\otimes} e\left(\frac{H\delta}{2};\theta\right) + e\left(-\frac{H\delta}{2};\theta\right)\bar{\otimes} J_\pm^\theta \\
&\varepsilon(J_\pm^\theta) = \varepsilon(H) = 0 \qquad \varepsilon(\mathbf{1}) = \mathbf{1} \\
&S(H) = -H \qquad S(J_\pm^\theta) = -e(\pm\delta;\theta)J_\pm^\theta \qquad S(\mathbf{1}) = \mathbf{1} \\
&\Delta\left(e\left(\frac{H\delta}{2};\theta\right)\right) = e\left(\frac{H\delta}{2};\theta\right)\bar{\otimes} e\left(\frac{H\delta}{2};\theta\right).
\end{aligned} \tag{3.16}$$

The following axioms are then satisfied,

$$\begin{aligned}
&(\mathrm{id}\bar{\otimes}\Delta)\Delta = (\Delta\bar{\otimes}\mathrm{id})\Delta \\
&M(\mathrm{id}\bar{\otimes}S)\Delta = m(S\bar{\otimes}\mathrm{id})\Delta = \mathrm{i}\circ\varepsilon \\
&(\varepsilon\bar{\otimes}\mathrm{id})\Delta = (\mathrm{id}\bar{\otimes}\varepsilon)\Delta = \mathrm{id}
\end{aligned} \tag{3.17}$$

with the coproduct Δ: $\mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2))\bar{\otimes}\mathcal{U}_q^r(sl(2))$, counit ε: $\mathcal{U}_q^r(sl(2)) \to \mathbb{C}[\theta]$, the antipode S: $\mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2))$ and m: $\mathcal{U}_q^r(sl(2))\bar{\otimes}\mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2))$, where the operations Δ, S and ε only act on H and $J_\pm^{(k)}$.

Let us now turn to some specific examples.

Example 1. The $r = 0$ case is characterized by

$$\theta = 0 \qquad J_\pm^\theta = J_\pm^{(0)}$$

and

$$[H, J_\pm^\theta] = \pm 2J_\pm^\theta \qquad [J_+^\theta, J_-^\theta] = H. \tag{3.18}$$

Thus, the $\mathcal{U}_q^0(sl(2))$ algebra is nothing but $sl(2)$, endowed as usual with

$$\Delta(H) = \mathbf{1} \otimes H + H \otimes \mathbf{1}$$
$$\Delta(J_\pm^\theta) = J_\pm^\theta \otimes \mathbf{1} + \mathbf{1} \otimes J_\pm^\theta \qquad \text{etc.}$$

Example 2. The $r = 1$ case is characterized by

$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad J_\pm^\theta = \begin{pmatrix} J_\pm^{(0)} & 0 \\ \delta J_\pm^{(1)} & J_\pm^{(0)} \end{pmatrix}$$

and the $sl(2)$ algebra (3.18) but now supplemented by a non-cocommutative coproduct

$$\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H$$
$$\Delta(J_\pm^\theta) = J_\pm^\theta \bar{\otimes} \left(\mathbf{1} + \frac{\theta\delta}{2} H\right) + \left(\mathbf{1} - \frac{\theta\delta}{2} H\right) \bar{\otimes} J_\pm^\theta. \tag{3.19}$$

Example 3. When $r = 2$, i.e.

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad J_\pm^\theta = \begin{pmatrix} J_\pm^{(0)} & 0 & 0 \\ \delta J_\pm^{(1)} & J_\pm^{(0)} & 0 \\ \delta^2 J_\pm^{(2)} & \delta J_\pm^{(1)} & J_\pm^{(0)} \end{pmatrix}$$

we obtain

$$[H, J_\pm^\theta] = \pm 2J_\pm^\theta \qquad [J_+^\theta, J_-^\theta] = H + \theta^2 \frac{\delta^3}{3!} (H^3 - H). \tag{3.20}$$

The coproduct is given by

$$\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H \tag{3.21}$$
$$\Delta(J_\pm^\theta) = J_\pm^\theta \bar{\otimes} \left(\mathbf{1} + \frac{\theta\delta}{2} H + \frac{(\theta\delta)^2}{8} H^2\right) + \left(\mathbf{1} - \frac{\theta\delta}{2} H + \frac{(\theta\delta)^2}{8} H^2\right) \bar{\otimes} J_\pm^\theta.$$

Such a structure is discussed in [7] in connection with the Higgs algebra, characterized by

$$[H, J_\pm] = \pm 2J_\pm \qquad [J_+, J_-] = H + cH^3 \tag{3.22}$$

c being an arbitrary constant. This algebra is of special interest as it appeared in the study of the harmonic oscillator and the Kepler problem in a two-dimensional curved space [8].

Example 4. The $r \to \infty$ case (θ is equivalent to a *real* variable) is characterized by

$$\begin{aligned} J_{\pm}^{\theta} &= \sum_{m=0}^{\infty} J_{\pm}^{(m)} \delta^m \theta^m \\ &= \sum_{m=0}^{\infty} J_{\pm}^{(m)} \zeta^m \\ J_{\pm}^{\theta} : &= \tilde{J}_{\pm} \end{aligned} \tag{3.23}$$

and

$$[H, \tilde{J}_{\pm}] = \pm 2 \tilde{J}_{\pm} \qquad [\tilde{J}_{+}, \tilde{J}_{-}] = \frac{e^{\zeta H} - e^{-\zeta H}}{e^{\zeta} - e^{-\zeta}} \tag{3.24}$$

where $\zeta = \theta\delta$. We thus recover the Drinfeld–Jimbo structure $\mathcal{U}_{e^{\zeta}}(sl(2))$ as a particular case of $\mathcal{U}_q^{\infty}(sl(2))$.

The same embedding is also present at the level of the Hopf structure with

$$\begin{aligned} &\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H \\ &\Delta(\tilde{J}_{\pm}) = \tilde{J}_{\pm} \otimes e^{\zeta H/2} + e^{-\zeta H/2} \otimes \tilde{J}_{\pm} \\ &\varepsilon(\tilde{J}_{\pm}) = \varepsilon(H) = 0 \\ &S(H) = -H \qquad S(\tilde{J}_{\pm}) = -e^{\pm\zeta} \tilde{J}_{\pm}. \end{aligned} \tag{3.25}$$

4. The $\mathcal{U}_{q_1,q_2}^{r_1,r_2}(sl(2))$ algebra

Let us now introduce, for example, two real paragrassmannian variables θ_1 and θ_2 of order r_1 and r_2, respectively, i.e.

$$\theta_1^{r_1+1} = 0 \qquad \theta_2^{r_2+1} = 0 \qquad \theta_1\theta_2 + \theta_2\theta_1 = 0. \tag{4.1}$$

Using the Campbell–Baker–Hausdorff expansion

$$(\exp A)(\exp B) = \exp C \tag{4.2}$$

where

$$\begin{aligned} &C = A + B + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{(m+1)!} (\mathrm{ad}A)^m(B) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+1)!} (\mathrm{ad}B)^m(A) \\ &(\mathrm{ad}A)^m(B) = [A, [A, \ldots, [A, B]] \cdots] \\ &(\mathrm{ad}B)^m(A) = [B, [B, \ldots, [B, A]] \cdots] \end{aligned} \tag{4.3}$$

we propose to define

$$J_{\pm}^{(\theta_1,\theta_2)} = \sum_{m=0}^{\infty} \theta^m J_{\pm}^{(m)} \tag{4.4}$$

where

$$\begin{aligned} &\theta = \theta_1\delta_1 + \theta_2\delta_2 + \frac{1}{2} \sum_{m=1}^{r_1} \frac{\delta_2 \delta_1^m 2^m}{(m+1)!} \theta_1^m \theta_2 + \frac{1}{2} \sum_{m=1}^{r_2} \frac{\delta_1 \delta_2^m 2^m}{(m+1)!} \theta_1 \theta_2^m \\ &\exp(\theta_1\delta_1) \exp(\theta_2\delta_2) = \exp\theta. \end{aligned} \tag{4.5}$$

Using (4.4), we deduce that

$$\begin{aligned} &[J_{+}^{(\theta_1,\theta_2)}, J_{-}^{(\theta_1,\theta_2)}] = \frac{e(H\delta_1;\theta_1)e(H\delta_2;\theta_2) - e(-H\delta_2;\theta_2)e(-H\delta_1;\theta_1)}{e(\delta_1;\theta_1)e(\delta_2;\theta_2) - e(-\delta_2;\theta_2)e(-\delta_1;\theta_1)} \\ &[H, J_{\pm}^{(\theta_1,\theta_2)}] = \pm 2 J_{\pm}^{(\theta_1,\theta_2)}. \end{aligned} \tag{4.6}$$

The algebra $\{J_\pm^{(\theta_1,\theta_2)}, H\}$ described by the commutation relations (4.6) is just the quantization of $sl(2)$ with two paragrassmannian variables and is denoted by $\mathcal{U}_{\delta_1,\delta_2}^{\theta_1,\theta_2}(sl(2))$. The $\mathcal{U}_{\delta_1,\delta_2}^{\theta_1,\theta_2}(sl(2))$ algebra is equipped with the following Hopf structure

$$\begin{aligned}
&\Delta(H) = H\otimes \mathbf{1} + \mathbf{1}\otimes H\\
&\Delta(J_\pm^{(\theta_1,\theta_2)}) = J_\pm^{(\theta_1,\theta_2)}\bar{\otimes} e\left(\frac{H\delta_1}{2};\theta_1\right)e\left(\frac{H\delta_2}{2};\theta_2\right) + e\left(-\frac{H\delta_2}{2};\theta_2\right)e\left(-\frac{H\delta_1}{2};\theta_1\right)\bar{\otimes} J_\pm^{(\theta_1,\theta_2)}\\
&\varepsilon(J_\pm^{(\theta_1,\theta_2)}) = \varepsilon(H) = 0 \qquad \varepsilon(\mathbf{1}) = \mathbf{1}\\
&S(H) = -H \qquad S(\mathbf{1}) = \mathbf{1}\\
&S(J_\pm^{(\theta_1,\theta_2)}) = -e\left(\frac{H\delta_1}{2};\theta_1\right)e\left(\frac{H\delta_2}{2};\theta_2\right)J_\pm^{(\theta_1,\theta_2)} e\left(-\frac{H\delta_2}{2};\theta_2\right)e\left(-\frac{H\delta_1}{2};\theta_1\right)\\
&\Delta\left(e\left(\frac{H\delta_1}{2};\theta_1\right)e\left(\frac{H\delta_2}{2};\theta_2\right)\right) = e\left(\frac{H\delta_1}{2};\theta_1\right)e\left(\frac{H\delta_2}{2};\theta_2\right)\bar{\otimes} e\left(\frac{H\delta_1}{2};\theta_1\right)e\left(\frac{H\delta_2}{2};\theta_2\right).
\end{aligned} \tag{4.7}$$

5. Connection with some nonlinear algebras

Let us take the following restriction in $\mathcal{U}_q^r(sl(2))$,

$$q = 1 \qquad \text{i.e. } q = \mathrm{e}^{2\pi \mathrm{i} n} \tag{5.1}$$

where n characterizes the Riemann branch. We have

$$e(2\pi\mathrm{i}n\Omega;\theta) = \cos(2\pi n\Omega;\theta) + \mathrm{i}\sin(2\pi n\Omega;\theta) \tag{5.2}$$

with

$$\begin{aligned}
\cos(x;\theta) &= \sum_{k=0}^{[r/2]}(-1)^k\frac{x^{2k}\theta^{2k}}{(2k)!}\\
\sin(x;\theta) &= \sum_{k=0}^{(r-1)/2-\frac{1}{2}(1+(-1)^r)}(-1)^k\frac{x^{2k+1}\theta^{2k+1}}{(2k+1)!}.
\end{aligned} \tag{5.3}$$

Thus, the commutation relations are written as

$$[H, J_\pm^\theta] = \pm 2J_\pm^\theta \qquad [J_+^\theta, J_-^\theta] = \frac{\sin(2\pi nH;\theta)}{\sin(2\pi n;\theta)}. \tag{5.4}$$

When $n\to\infty$, we deduce

$$\lim_{n\to\infty}\frac{\sin(2\pi nH;\theta)}{\sin(2\pi n;\theta)} = H^{r-\frac{1}{2}(1+(-1)^r)} \tag{5.5}$$

and

$$[H, J_\pm^\theta] = \pm 2J_\pm^\theta \qquad [J_+^\theta, J_-^\theta] = H^{r-\frac{1}{2}(1+(-1)^r)} \tag{5.6}$$

the deformation being a nonlinear one.

Now, if we take in $\mathcal{U}_{q_1,q_2}^{r_1,r_2}(sl(2))r_2\to\infty$ and $\delta_1 = 2\pi\mathrm{i}n(n\to\infty)$, we deduce the following nonlinear algebra

$$[J_+, J_-] = \frac{H^{r_1}q^H - (-1)^{r_1}q^{-H}H^{r_1}}{q-(-1)^{r_1}q^{-1}} \qquad [H, J_\pm] = \pm J_\pm. \tag{5.7}$$

6. Conclusion

We have proposed new deformed structure $\mathcal{U}_q^r(sl(2))$ and $\mathcal{U}_{q_1,q_2}^{r_1,r_2}(sl(2))$ obtained by paragrassmannian deformation. When the order of the paragrassmannian variable goes to infinity, we recover the Drinfeld–Jimbo scheme of deformation.

It has also to be noticed that our proposal points out two different Hopf structures for the same deformed algebra. In particular, $sl(2)$ can be associated with a cocommutative coproduct ($r = 0$) or a non-commutative one ($r = 1$). Then it is possible to get a new $\mathcal{R}$-matrix given by

$$\begin{aligned}\mathcal{R}_\theta &= 1 \otimes 1 + \delta\theta(J_- \otimes J_+ - J_+ \otimes J_-) \\ &= U_\theta U_{-\theta}^+\end{aligned} \tag{6.1}$$

where

$$U_\theta = 1 \otimes 1 + \tfrac{1}{2}\delta\theta(J_- \otimes J_+ - J_+ \otimes J_-) \tag{6.2}$$

by requiring

$$U_\theta \Delta_{r=0}(a) = \Delta_{r=1}(a) U_\theta \tag{6.3}$$

for any a belonging to $sl(2)$. It is also noticed that this matrix $\mathcal{R}_\theta$ satisfies the Yang–Baxter equation. Thus it is the first solution, to our knowledge, depending on a paragrassmannian variable.

We would like to note that the $r = 2$ case is a particular interesting one, as already mentioned. It is the first case where the deformation is present at the level of the algebra and these deformations are nonlinear ones in the sense of Roček. We have thus defined ad-hoc coproducts, counits and antipodes for such deformations being of physical interest.

Finally, the restriction of the values of the parameters of the deformation gives some nonlinear algebra as particular cases.

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